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ADDITIONAL REDUCTIONS IN THE K-CONSTRAINED MODIFIED KP HIERARCHY

O. CHVARTATSKYI AND YU. SYDORENKO

Abstract. Additional reductions in the modified k-constrained KP hierarchy are proposed. As a result we obtain generalizations of Kaup-Broer system, Korteweg-de Vries equation and a modification of Korteweg-de Vries equation that belongs to the modified k-constrained KP hierarchy. We also propose solution generating technique based on binary Darboux transformations for the obtained equations.

1. Introduction

The algebraic constructions of the well-known Kyoto group [1], which are called the Sato theory, play an important role in the contemporary theory of nonlinear integrable systems of mathematical and theoretical physics. The leading place in these investigations is occupied by the theory of equations of Kadomtsev-Petviashvili type (KP hierarchy) and their generalizations and applications [1–3].

One of known generalizations of the KP hierarchy arise as a result of k-symmetry constraints (so-called k-cKP hierarchy) that were investigated in [4–8]. k-cKP hierarchy are closely connected with so-called KP equation with self-consistent sources (KPSCS) [9–12]. Multicomponent k-constraints of the KP hierarchy were introduced in [13] and investigated in [14–18]. This extension of k-cKP hierarchy contains vector (multicomponent) generalizations of such physically relevant systems like the nonlinear Schrödinger equation, the Yajima-Oikawa system, a generalization of the Boussinesq equation, and the Melnikov system.

The modified k-constrained KP (k-cmKP) hierarchy was proposed in [19,20]. It contains, for example, the vector Chen-Lee-Liu, the modified KdV (mKdV) equation and their multi-component extensions. The k-cmKP hierarchy and dressing methods for it via integral transformations were investigated in [21–23]

In [24,25] (2+1)-dimensional extensions of the k-cKP hierarchy ((2+1)-dimensional k-cKP hierarchy) were introduced and dressing methods via differential transformations were investigated. Some systems of this hierarchy were investigated via binary Darboux transformations in [22, 23]. This hierarchy was also rediscovered recently in [26, 27]. Matrix generalizations of (2+1)-dimensional k-cKP hierarchy were considered in [28, 29].

In this paper our aim is to consider additional reductions of the k-cmKP hierarchy that lead to new generalizations of well-known integrable systems. We also investigated dressing methods for the obtained systems via integral transformations that arise from Binary Darboux Transformations (BDT).

Key words and phrases. solitons, binary Darboux transformation, modified constrained Kadomtsev-Petviashvili hierarchy,Grammian solutions.

This work is organized as follows. In Section 2 we present a short survey of results on constraints for the KP hierarchies including the k-cmKP hierarchy. In Section 3 we investigate Lax representations obtained as a result of additional reductions in the k-cmKP hierarchy and corresponding nonlinear systems. Section 4 presents results on dressing methods for Lax pairs obtained in Section 3. In the final section, we discuss the obtained results and mention problems for further investigations.

2. Symmetry constraints of the KP hierarchy

Let us recall some basic objects and notations concerning KP hierarchy, modified KP hierarchy, their multicomponent k-constraints and their (2+1)-extensions. A Lax representation of the KP hierarchy is given by

$$L_{t_n} = [B_n, L], \qquad n \geq 1, \tag{1}$$

where $L = D + U_1 D^{-1} + U_2 D^{-2} + \ldots$ is a scalar pseudodifferential operator, $t_1 := x$, $D := \frac{\partial}{\partial x}$, and $B_n := (L^n)_+ := (L^n)_{\geq 0} = D^n + \sum_{i=0}^{n-2} u_i D^i$ is the differential operator part of L^n. The consistency condition (zero-curvature equations), arising from the commutativity of flows (1), is

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0. \tag{2}$$

Let B_n^τ denote the formal transpose of B_n, i.e. $B_n^\tau := (-1)^n D^n + \sum_{i=0}^{n-2}(-1)^i D^i u_i^\top$, where $^\top$ denotes the matrix transpose. We will use curly brackets to denote the action of an operator on a function whereas, for example, $B_n\, q$ means the composition of the operator B_n and the operator of multiplication by the function q. The following formula holds for $B_n q$ and $B_n\{q\}$: $B_n\{q\} := (B_n q)_{=0} = B_n q - (B_n q)_{>0}$. In the case $k = 2$, $n = 3$ formula (8) presents a Lax pair for the Kadomtsev-Petviahvili equation. Its Lax pair was obtained in [30] (see also [31]).

The multicomponent k-constraints of the KP hierarchy is given by [13]

$$L_{t_n} = [B_n, L], \tag{3}$$

with the k-symmetry reduction

$$L_k := L^k = B_k + \sum_{i=1}^{m} \sum_{j=1}^{m} q_i m_{ij} D^{-1} r_j = B_k + \mathbf{q} \mathcal{M}_0 D^{-1} \mathbf{r}^\top, \tag{4}$$

where $\mathbf{q} = (q_1, \ldots, q_m)$ and $\mathbf{r} = (r_1, \ldots, r_m)$ are vector functions, $\mathcal{M}_0 = (m_{ij})_{i,j=1}^m$ is a constant $m \times m$ matrix. In the scalar case ($m = 1$) we obtain k-constrained KP hierarchy [4–8]. The hierarchy given by (3)-(4) admits the Lax representation (here $k \in \mathbf{N}$ is fixed):

$$[L_k, M_n] = 0, \;\; L_k = B_k + \mathbf{q}\mathcal{M}_0 D^{-1} \mathbf{r}^\top, \quad M_n = \partial_{t_n} - B_n. \tag{5}$$

Lax equation (5) is equivalent to the following system:

$$[L_k, M_n]_{\geq 0} = 0, \;\; M_n\{\mathbf{q}\} = 0, \;\; M_n^\tau\{\mathbf{r}\} = 0. \tag{6}$$

Below we will also use the formal adjoint $B_n^* := \bar{B}_n^\tau = (-1)^n D^n + \sum_{i=0}^{n-2}(-1)^i D^i u_i^*$ of B_n, where * denotes the Hermitian conjugation (complex conjugation and transpose).

For $k = 1$, the hierarchy given by (6) is a multi-component generalization of the AKNS hierarchy. For $k = 2$ and $k = 3$, one obtains vector generalizations of the Yajima-Oikawa and Melnikov [9–11] hierarchies, respectively. An essential

extension of the k-cKP hierarchy is its (2+1)-dimensional generalization introduced in [24, 25] and rediscovered in [26, 27].

In [19, 20], a k-constrained modified KP (k-cmKP) hierarchy was introduced and investigated. Dressing methods for k-cmKP hierarchy under additional D-Hermitian reductions were also investigated in [21, 22]. At first we recall the definition of the modified KP hierarchy. A Lax representation of this hierarchy is given by

$$L_{t_n} = [B_n, L], \qquad n \geq 1, \tag{7}$$

where $L = D + U_0 + U_1 D^{-1} + U_2 D^{-2} + \ldots$ and $B_n := (L^n)_{>0} := D^n + \sum_{i=1}^{n-1} u_i D^i$ is the purely differential operator part of L^n. The consistency condition arising from the commutativity of flows (7), is

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0. \tag{8}$$

The multicomponent k-constraints of the modified KP hierarchy are given by the operator equation:

$$L_{t_n} = [B_n, L], \tag{9}$$

with the k-symmetry reduction

$$L_k := L^k = B_k - \sum_{i=1}^{m}\sum_{j=1}^{m} q_i m_{ij} D^{-1} r_j D = B_k - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D, \tag{10}$$

where $\mathbf{q} = (q_1, \ldots, q_m)$ and $\mathbf{r} = (r_1, \ldots, r_m)$ are vector functions, $\mathcal{M}_0 = (m_{ij})_{i,j=1}^m$ is a constant $m \times m$ matrix. The hierarchy (9)-(10) admits the Lax representation (here $k \in \mathbf{N}$ is fixed):

$$\begin{aligned} &[L_k, M_n] = 0, \;\; L_k = B_k - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D, \\ &M_n = \alpha_n \partial_{t_n} - B_n, \;\; B_n = D^n + \textstyle\sum_{i=1}^{n-1} u_i D^i. \end{aligned} \tag{11}$$

We can rewrite the Lax pair (11) in the following way:

$$\begin{aligned} &[L_k, M_n] = 0, \;\; L_k = B_k - \mathbf{q}\mathcal{M}_0 \mathbf{r}^\top + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}_x^\top, \\ &M_n = \alpha_n \partial_{t_n} - B_n, \;\; B_n = D^n + \textstyle\sum_{i=1}^{n-1} u_i D^i. \end{aligned} \tag{12}$$

From Lax representation for k-cKP hierarchy (5)-(6) and representation (12) we come to conclusion that equation $[L_k, M_n] = 0$ in (11) is equivalent to the following system: $[L_k, M_n]_{>0} = 0$, $M_n\{\mathbf{q}\} = 0$, $(M_n^\tau)\{\mathbf{r}_x\} = 0$ ($[L_k, M_n]_{=0} = 0$ since $[L_k, M_n]\{1\} = 0$). We can rewrite the last equation in the following form: $(D^{-1}M_n^\tau D)\{\mathbf{r}\} = 0$ to keep the order of differentiation equal to n. As a result we obtain:

$$[L_k, M_n]_{>0} = 0, \;\; M_n\{\mathbf{q}\} = 0, \;\; (D^{-1}M_n^\tau D)\{\mathbf{r}\} = 0. \tag{13}$$

The hierarchy (11) contains vector generalizations of the Chen-Lee-Liu ($k = 1$), the modified multi-component Yajima-Oikawa ($k = 2$) and Melnikov ($k = 3$) hierarchies. Consider some equations that can be obtained from (11) under certain choice of k and n (see [23]).

(1) $k = 1$, $n = 2$. Then (11) becomes

$$L_1 = D - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D, \;\; M_2 = \alpha_2\partial_{t_2} - D^2 + 2\mathbf{q}\mathcal{M}_0\mathbf{r}^\top D. \tag{14}$$

In this case equation (13) becomes the following system:

$$\alpha_2\mathbf{q}_{t_2} - \mathbf{q}_{xx} + 2\mathbf{q}\mathcal{M}_0\mathbf{r}^\top\mathbf{q}_x = 0, \; \alpha_2\mathbf{r}_{t_2}^\top + \mathbf{r}_{xx}^\top + 2\mathbf{r}_x^\top\mathbf{q}\mathcal{M}_0\mathbf{r}^\top = 0. \tag{15}$$

Under additional Hermitian conjugation reduction: $\alpha_2 = i$, $\mathcal{M}_0 = -\mathcal{M}_0^*$, $\mathbf{r}^\top = \mathbf{q}^*$ ($L_1^* = -D^{-1}L_1D$, $M_2^* = D^{-1}M_2D$) in (15) we obtain the Chen-Lee-Liu equation:

$$i\mathbf{q}_{t_2} - \mathbf{q}_{xx} + 2\mathbf{q}\mathcal{M}_0\mathbf{q}^*\mathbf{q}_x = 0. \tag{16}$$

(2) $k = 1$, $n = 3$. In this case (11) takes the form:

$$\begin{aligned} &L_1 = D - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D, \\ &M_3 = \alpha_3\partial_{t_3} - D^3 + 3\mathbf{q}\mathcal{M}_0\mathbf{r}^\top D^2 + 3[\mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top - (\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)^2]D, \end{aligned} \tag{17}$$

and equations (13) read:

$$\begin{aligned} \alpha_3\mathbf{q}_{t_3} &= \mathbf{q}_{xxx} - 3(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)\mathbf{q}_{xx} - 3(\mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top - (\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)^2)\mathbf{q}_x, \\ \alpha_3\mathbf{r}^\top_{t_3} &= \mathbf{r}^\top_{xxx} + 3\mathbf{r}^\top_{xx}(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top) + 3\mathbf{r}^\top_x(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top_x + (\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)^2). \end{aligned} \tag{18}$$

After reduction of Hermitian conjugation: $\alpha_3 = 1$, $\mathbf{r}^\top = \mathbf{q}^*$, $\mathcal{M}_0 = -\mathcal{M}_0^*$ ($L_1^* = -D^{-1}L_1D$, $M_3^* = -D^{-1}M_3D$) (18) becomes:

$$\mathbf{q}_{t_3} = \mathbf{q}_{xxx} - 3(\mathbf{q}\mathcal{M}_0\mathbf{q}^*)\mathbf{q}_{xx} - 3(\mathbf{q}_x\mathcal{M}_0\mathbf{q}^* - (\mathbf{q}\mathcal{M}_0\mathbf{q}^*)^2)\mathbf{q}_x. \tag{19}$$

(3) $k = 2$, $n = 2$. After additional reduction in (11): $\alpha_2 = i$, $u_1 := iu$, $u = u(x, t_2) \in \mathbb{R}$, $\mathcal{M}_0 = \mathcal{M}_0^*$ Lax pair in (13) reads:

$$[L_2, M_2] = 0,\ L_2 = D^2 + iuD - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^* D,\ M_2 = i\partial_{t_2} - D^2 - iuD,$$

and equation (13) becomes the modified Yajima-Oikawa equation:

$$i\mathbf{q}_{t_2} = \mathbf{q}_{xx} + iu\mathbf{q}_x, \quad u_{t_2} = 2(\mathbf{q}\mathcal{M}_0\mathbf{q}^*)_x.$$

In the next section we will introduce additional reductions in Chen-Lee-Liu hierarchy. As a result we will obtain generalizations of the Kaup-Broer system, Korteweg-de Vries equation, modified Korteweg-de Vries equation and their scalar coupled versions.

3. Additional reductions in the modified k-constrained KP hierarchy

For further convenience let us make a change in formulae (11):

$$\mathbf{q} \to \tilde{\mathbf{q}},\ \mathbf{r} \to \tilde{\mathbf{r}},\ \mathcal{M}_0 \to \tilde{\mathcal{M}}_0. \tag{20}$$

After the change (20) the hierarchy (11) reads:

$$\begin{aligned} &[L_k, M_n] = 0,\ L_k = B_k - \tilde{\mathbf{q}}\tilde{\mathcal{M}}_0 D^{-1}\tilde{\mathbf{r}}^\top D, \quad M_n = \alpha_n\partial_{t_n} - B_n, \\ &B_n = D^n + \textstyle\sum_{i=1}^{n-1} u_i D^i. \end{aligned} \tag{21}$$

Let us make the additional reduction in (21):

$$\begin{aligned} &\tilde{\mathbf{q}} := (q_1, \ldots, q_m, -v - \beta D^{-1}\{u\}, 1) = (\mathbf{q}, -v - \beta D^{-1}\{u\}, 1), \\ &\tilde{\mathcal{M}}_0 = \begin{pmatrix} \mathcal{M}_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \tilde{\mathbf{r}} := (r_1, \ldots, r_m, 1, \beta D^{-1}\{u\}) = (\mathbf{r}, 1, \beta D^{-1}\{u\}), \end{aligned} \tag{22}$$

where $\mathcal{M}_0$ is $(m \times m)$-constant matrix, $\mathbf{q}$ and $\mathbf{r}$ are m-component vectors, u and v are scalar functions, $\beta \in \mathbb{R}$, $D^{-1}\{u\}$ denotes indefinite integral of the function u with respect to x. After reduction (22) k-cmKP hierarchy (21) takes the form:

$$\begin{aligned} &[L_k, M_n] = 0,\ L_k = B_k - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D + v + \beta D^{-1}u,\ M_n = \alpha_n\partial_{t_n} - B_n, \\ &B_n = D^n + \textstyle\sum_{i=1}^{n-1} u_i D^i. \end{aligned} \tag{23}$$

In the following subsections we will investigate hierarchy (23) in case $k = 1$.

3.1. **Reductions of the Chen-Lee-Liu system.** Let us put $k=1$, $n=2$. Then Lax pair (23) becomes:

$$\begin{gathered}[L_1, M_2]=0,\ L_1 = D - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D + \beta D^{-1}u + v,\\ M_2 = \alpha_2\partial_{t_2} - D^2 + 2(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)D.\end{gathered} \tag{24}$$

A system that corresponds to equation (24) has the form:

$$\begin{gathered}\alpha_2\mathbf{q}_{t_2} = \mathbf{q}_{xx} - 2(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)\mathbf{q}_x, \quad \alpha_2\mathbf{r}^\top_{t_2} = -\mathbf{r}^\top_{xx} - 2\mathbf{r}^\top_x(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v),\\ \alpha_2 u_{t_2} + u_{xx} + 2\left(u(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)\right)_x = 0,\\ -\alpha_2 v_{t_2} + 2\beta u_x + v_{xx} - 2\left(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v\right)v_x = 0.\end{gathered} \tag{25}$$

Consider additional reductions of Lax pair (24) and system (25).

(1) Assume that $\mathcal{M}_0 = -\mathcal{M}_0^*$, $\mathbf{r}^\top = \mathbf{q}^*$, $v = -2i\mathrm{Im}(\beta D^{-1}\{u\})$ ($L_1^* = -DL_1D^{-1}$, $M_2^* = DM_2D^{-1}$). Then equation (25) takes the form:

$$\begin{gathered}\alpha_2\mathbf{q}_{t_2} = \mathbf{q}_{xx} - 2(2i\mathrm{Im}(\beta D^{-1}\{u\}) + \mathbf{q}\mathcal{M}_0\mathbf{q}^*)\mathbf{q}_x,\\ \alpha_2 u_{t_2} + u_{xx} + 2\left(u(2i\mathrm{Im}(\beta D^{-1}\{u\}) + \mathbf{q}\mathcal{M}_0\mathbf{q}^*)\right)_x = 0.\end{gathered}$$

(2) Let us put $\mathcal{M}_0 = 0$ in operators L_1 and M_2: $L_1 = D + \beta D^{-1}u + v$, $M_2 = \alpha_2\partial_{t_2} - D^2 - 2vD$. Then equation (25) becomes the Kaup-Broer system:

$$\alpha_2 u_{t_2} + u_{xx} - 2(uv)_x = 0,\ -\alpha_2 v_{t_2} + 2\beta u_x + v_{xx} + 2vv_x = 0. \tag{26}$$

In case $u=0$ in (26) we obtain the Burgers equation: $-\alpha_2 v_{t_2} + v_{xx} - vv_x = 0$.

(3) Consider the case $u=0$ in operators L_1 and M_2 (25): $L_1 = D - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D + v$, $M_2 = \alpha_2\partial_{t_2} - D^2 + 2(v + \mathbf{q}\mathcal{M}_0\mathbf{r}^\top)D$. Then (25) reads:

$$\begin{gathered}\alpha_2\mathbf{q}_{t_2} = \mathbf{q}_{xx} - 2(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)\mathbf{q}_x, \quad \alpha_2\mathbf{r}^\top_{t_2} = -\mathbf{r}^\top_{xx} - 2\mathbf{r}^\top_x(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v),\\ -\alpha_2 v_{t_2} + v_{xx} - \left(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v\right)v_x = 0.\end{gathered}$$

3.2. **Reductions of the modification of Korteweg-de Vries system (19).** Now let us consider the hierarchy (23) in case $k=1$, $n=3$. Then its Lax pair L_1, M_3 in (23) reads:

$$\begin{gathered}[L_1, M_3]=0,\ L_1 = D - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top D + \beta D^{-1}u + v,\ M_3 = \alpha_3\partial_{t_3} - D^3 -\\ -3(v - \mathbf{q}\mathcal{M}_0\mathbf{r}^\top)D^2 - 3\left((\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)^2 - \mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top + \beta u + v_x\right)D.\end{gathered} \tag{27}$$

Commutator equation in (27) is equivalent to the system:

$$\begin{gathered}-\alpha_3 v_{t_3} + v_{xxx} + 3vv_{xx} + 3v^2v_x + 3v_x^2 + 6\beta(uv)_x +\\ +3\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)^2 - \mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top\right\}v_x - 3\mathbf{q}\mathcal{M}_0\mathbf{r}^\top v_{xx} -\\ -6\mathbf{q}\mathcal{M}_0\mathbf{r}^\top vv_x - 3\beta(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top u)_x - 3\beta\mathbf{q}\mathcal{M}_0\mathbf{r}^\top u_x = 0,\\ \alpha_3\mathbf{q}_{t_3} = \mathbf{q}_{xxx} + 3(v - \mathbf{q}\mathcal{M}_0\mathbf{r}^\top)\mathbf{q}_{xx} +\\ +3\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)^2 - \mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top + v_x + \beta u\right\}\mathbf{q}_x,\\ \alpha_3\mathbf{r}^\top_{t_3} = \mathbf{r}^\top_{xxx} - 3\left(\mathbf{r}^\top_x\left(v - \mathbf{q}\mathcal{M}_0\mathbf{r}^\top\right)\right)_x +\\ +3\mathbf{r}^\top_x\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)^2 - \mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top + v_x + \beta u\right\},\\ \alpha_3 u_{t_3} = u_{xxx} - 3\left(u(v - \mathbf{q}\mathcal{M}_0\mathbf{r}^\top)\right)_{xx} +\\ +3\left(u\left((\mathbf{q}\mathcal{M}_0\mathbf{r}^\top - v)^2 - \mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top + v_x + \beta u\right)\right)_x.\end{gathered} \tag{28}$$

Consider additional reductions in Lax pair (27) and corresponding system (28).

(1) Assume that $v=-2i\mathrm{Im}(\beta D^{-1}\{u\})$, $\mathbf{q}^*=\mathbf{r}^\top$, $u\in\mathbb{R}$, $\mathcal{M}_0=-\mathcal{M}_0^*$ ($L_1^*=-DL_1D^{-1}$, $M_3^*=-DM_3D^{-1}$). Then system (28) takes the form:

$$\begin{gathered}\alpha_3\mathbf{q}_{t_3}=\mathbf{q}_{xxx}-3(2iIm(\beta u)+\mathbf{q}\mathcal{M}_0\mathbf{q}^*)\mathbf{q}_{xx}+\\ +3\left((\mathbf{q}\mathcal{M}_0\mathbf{q}^*+2i\mathrm{Im}(\beta u))^2\ -\mathbf{q}_x\mathcal{M}_0\mathbf{q}^*+\beta u-2i\mathrm{Im}(\beta u)\right)\mathbf{q}_x,\\ \alpha_3u_{t_3}=u_{xxx}+3\left\{u\left(2i\mathrm{Im}(\beta u)+\mathbf{q}\mathcal{M}_0\mathbf{q}^*\right)\right\}_{xx}+\\ +3\left(u\left\{(\mathbf{q}\mathcal{M}_0\mathbf{q}^*+2i\mathrm{Im}(\beta u))^2-\ \mathbf{q}_x\mathcal{M}_0\mathbf{q}^*+\ \beta u-2i\mathrm{Im}(\beta u)\right\}\right)_x.\end{gathered}\tag{29}$$

(a) Let us assume that in addition to reductions described in item 1 functions $\mathbf{q}$ and u with matrix $\mathcal{M}_0$ are real-valued (i.e., matrix $\mathcal{M}_0$ is skew-symmetric: $\mathcal{M}_0^\top=-\mathcal{M}_0$) and $v=0$. Then the scalar $\mathbf{q}\mathcal{M}_0\mathbf{q}^\top=0$ since $\mathbf{q}\mathcal{M}_0\mathbf{q}^\top=-(\mathbf{q}\mathcal{M}_0\mathbf{q}^\top)^\top$ and equation (29) reads:

$$\begin{gathered}\alpha_3\mathbf{q}_{t_3}=\mathbf{q}_{xxx}-3\mathbf{q}_x\mathcal{M}_0\mathbf{q}^\top\mathbf{q}_x+3\beta u\mathbf{q}_x,\\ \alpha_3u_{t_3}=u_{xxx}-3(u\mathbf{q}_x\mathcal{M}_0\mathbf{q}^\top)_x+6\beta uu_x.\end{gathered}\tag{30}$$

(2) Let us put $\mathcal{M}_0=0$ in operators L_1, M_3 (27):
$L_1=D+\beta D^{-1}u+v,\ M_3=\alpha_3\partial_{t_3}-D^3-3vD^2-3(v^2+v_x+\beta u)D.$
Then equation (28) takes the form:

$$\begin{gathered}-\alpha_3v_{t_3}+v_{xxx}+3vv_{xx}+3v^2v_x+3v_x^2+6\beta(uv)_x=0,\\ \alpha_3u_{t_3}=u_{xxx}-3(uv)_{xx}+3(u(v^2+v_x+\beta u))_x.\end{gathered}\tag{31}$$

(a) Under additional restrictions $v=-2i\mathrm{Im}(D^{-1}\{\beta u\})$ ($L_1^*=-DL_1D^{-1}$, $M_3^*=-DM_3D^{-1}$) in item 2 we obtain a complex generalization of the modified Korteweg-de Vries equation:

$$\begin{gathered}\alpha_3u_{t_3}=u_{xxx}+6i(uIm(D^{-1}\{\beta u\}))_{xx}+\\ +3(u(-4Im(D^{-1}\{\beta u\})^2-2i\mathrm{Im}(\alpha u)+\beta u))_x.\end{gathered}\tag{32}$$

In the real case ($\beta\in\mathbb{R}$, u is a real-valued function, $v=0$) operators L_1 and M_3 take the form: $L_1=D+\beta D^{-1}u$, $M_3=\beta\partial_t-D^3-3\beta uD$, and we obtain KdV equation in (32):

$$\alpha_3u_{t_3}=u_{xxx}+6\beta uu_x.\tag{33}$$

(3) Let us put $u=0$ in Lax pair (27): $L_1=D-\mathbf{q}\mathcal{M}_0D^{-1}\mathbf{r}^\top D+v$, $M_3=\alpha_3\partial_{t_3}-D^3-3(v-\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)D^2-3\left((\mathbf{q}\mathcal{M}_0\mathbf{r}^\top-v)^2-\mathbf{q}\mathcal{M}_0\mathbf{r}^\top+v_x\right)D$. Equation (28) becomes:

$$\begin{gathered}-\alpha_3v_{t_3}+v_{xxx}+3vv_{xx}+3v^2v_x+3v_x^2+\\ +3\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)^2-\mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top\right\}v_x-3\mathbf{q}\mathcal{M}_0\mathbf{r}^\top v_{xx}-6\mathbf{q}\mathcal{M}_0\mathbf{r}^\top vv_x=0,\\ \alpha_3\mathbf{q}_{t_3}=\mathbf{q}_{xxx}+3(v-\mathbf{q}\mathcal{M}_0\mathbf{r}^\top)\mathbf{q}_{xx}+\\ +3\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top-v)^2-\mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top+v_x\right\}\mathbf{q}_x,\\ \alpha_3\mathbf{r}^\top_{t_3}=\mathbf{r}^\top_{xxx}-3\left(\mathbf{r}_x^\top\left(v-\mathbf{q}\mathcal{M}_0\mathbf{r}^\top\right)\right)_x+\\ +3\mathbf{r}_x^\top\left\{(\mathbf{q}\mathcal{M}_0\mathbf{r}^\top-v)^2-\mathbf{q}_x\mathcal{M}_0\mathbf{r}^\top+v_x\right\}.\end{gathered}\tag{34}$$

4. Dressing methods for k-cmKP hierarchy

In this section our aim is to elaborate dressing methods for the k-cmKP hierarchy (11). At first we recall a main result from paper [35]. Let $1\times K$-matrix functions φ and ψ be solutions of linear problems with (2+1)-dimensional generalization of the operator L_k (4) wit more general differential part B_k:

$$\begin{gathered}L_k\{\varphi\}=\varphi\Lambda,\ L_k^\tau\{\psi\}=\psi\tilde{\Lambda},\ \Lambda,\tilde{\Lambda}\in Mat_{K\times K}(\mathbb{C}),\\ L_k=\beta_k\partial_{\tau_k}+B_k+\mathbf{q}\mathcal{M}_0D^{-1}\mathbf{r}^\top,\ B_k=\textstyle\sum_{j=0}^k u_jD^j.\end{gathered}\tag{35}$$

Introduce a binary Darboux transformation (BDT) in the following way:

$$W = I - \varphi\left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1} D^{-1}\psi^\top := I - \varphi\Delta^{-1}D^{-1}\psi^\top, \tag{36}$$

where C is a $K \times K$-constant nondegenerate matrix. The inverse operator W^{-1} has the form:

$$W^{-1} = I + \varphi D^{-1}\left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1}\psi^\top = I + \varphi D^{-1}\Delta^{-1}\psi^\top. \tag{37}$$

The following theorem is proven in [35].

Theorem 1. *[35] The operator $\hat{L}_k := WL_kW^{-1}$ obtained from L_k in (35) via BDT (36) has the form*

$$\begin{array}{l} \hat{L}_k := WL_kW^{-1} = \beta_k\partial_{\tau_k} + \hat{B}_k + \hat{\mathbf{q}}\mathcal{M}_0D^{-1}\hat{\mathbf{r}}^\top + \Phi\mathcal{M}D^{-1}\Psi^\top, \\ \hat{B}_k = \sum_{j=0}^k \hat{u}_jD^j, \end{array} \tag{38}$$

where

$$\begin{array}{l} \mathcal{M} = C\Lambda - \tilde{\Lambda}^\top C,\ \Phi = \varphi\Delta^{-1},\ \ \Psi = \psi\Delta^{-1,\top},\ \Delta = C + D^{-1}\{\psi^\top\varphi\}, \\ \hat{\mathbf{q}} = W\{\mathbf{q}\},\ \ \hat{\mathbf{r}} = W^{-1,\tau}\{\mathbf{r}\}. \end{array} \tag{39}$$

$\hat{u}_j$ are scalar coefficients depending on functions φ, ψ and u_i, $i = \overline{0,j}$. In particular,

$$\hat{u}_k = u_k, \quad \hat{u}_{k-1} = u_{k-1}, \ldots.$$

The exact forms of all the coefficients $\hat{u}_j$ can be found in [35].

Using the previous theorem we obtain the following result for (2+1)-generalization of operator L_k from the k-cmKP hierarchy (11):

Theorem 2. *Let $(1 \times K)$-vector functions φ and ψ satisfy linear problems:*

$$\begin{array}{c} L_k\{\varphi\} = \varphi\Lambda,\ L_k^\tau\{\psi\} = \psi\tilde{\Lambda},\ \ \Lambda, \tilde{\Lambda} \in Mat_{K\times K}(\mathbb{C}), \\ L_k = \beta_k\partial_{\tau_k} + B_k - \mathbf{q}\mathcal{M}_0D^{-1}\mathbf{r}^\top D,\ B_k = \sum_{i=1}^k u_iD^i. \end{array} \tag{40}$$

Then the operator $\hat{L}_k$ transformed via operator

$$W_m := w_0^{-1}W = w_0^{-1}\left(I - \varphi\Delta^{-1}D^{-1}\psi^\top\right) = I - \varphi\tilde{\Delta}^{-1}D^{-1}(D^{-1}\{\psi\})^\top D, \tag{41}$$

where

$$\begin{array}{l} w_0 = I - \varphi\Delta^{-1}D^{-1}\{\psi^\top\}, \tilde{\Delta} = -C + D^{-1}\{D^{-1}\{\psi^\top\}\varphi_x\}, \\ \Delta = C + D^{-1}\{\psi^\top\varphi\}, \end{array}$$

has the form:

$$\begin{array}{l} \tilde{L}_k := W_mL_kW_m^{-1} = \beta_k\partial_{\tau_k} + \tilde{B}_k - \tilde{\mathbf{q}}\mathcal{M}_0D^{-1}\tilde{\mathbf{r}}^\top D + \tilde{\Phi}\mathcal{M}D^{-1}\tilde{\Psi}^\top D, \\ \tilde{B}_k = \sum_{j=1}^k \tilde{u}_jD^j,\ \tilde{u}_k = u_k,\ \tilde{u}_{k-1} = u_{k-1} + ku_kw_0^{-1}w_{0,x}, \ldots, \end{array} \tag{42}$$

where

$$\begin{array}{l} \mathcal{M} = C\Lambda - \tilde{\Lambda}^\top C,\ \tilde{\Phi} = -W_m\{\varphi\}C^{-1} = \varphi\tilde{\Delta}^{-1}, \\ \tilde{\Psi} = D^{-1}\{W_m^{\tau,-1}\{\psi\}\}C^{-1,\top} = D^{-1}\{\psi\}\Delta^{-1,\top},\ \tilde{\mathbf{q}} = W_m\{\mathbf{q}\}, \\ \tilde{\mathbf{r}} = D^{-1}W_m^{-1,\tau}D\{\mathbf{r}\}, \tilde{\Delta} = -C + D^{-1}\{D^{-1}\{\psi^\top\}\varphi_x\}. \end{array} \tag{43}$$

Proof. Let us check that

$$w_0^{-1} = I - \varphi\tilde{\Delta}D^{-1}\{\psi^\top\},\ \tilde{\Delta} = -C + D^{-1}\{D^{-1}\{\psi^\top\}\varphi_x\}.$$

In order to do that we have to verify the equality $w_0w_0^{-1} = I$:

$$w_0w_0^{-1} = I - \varphi\Delta^{-1}D^{-1}\{\psi^\top\} - \varphi\tilde{\Delta}^{-1}D^{-1}\{\psi^\top\}$$

$$+\varphi\tilde{\Delta}^{-1}\left(C+D^{-1}\{\psi^\top\varphi\}-C+D^{-1}\{D^{-1}\{\psi^\top\}\varphi_x\}\right)\varphi\Delta^{-1}D^{-1}\{\psi^\top\}=I$$

Analogously can be verified that $w_0^{-1}w_0=I$. By Theorem 1 we obtain:

$$\begin{gathered}W_mL_kW_m^{-1}=w_0^{-1}W\left(\beta_k\partial_{\tau_k}+B_k-\mathbf{q}\mathcal{M}_0\mathbf{r}^\top+\mathbf{q}\mathcal{M}_0D^{-1}\mathbf{r}_x^\top\right)W^{-1}w_0=\\ \beta_k\partial_{\tau_k}+(W_mL_kW_m^{-1})_{\geq 0}-w_0^{-1}W\{\mathbf{q}\}\mathcal{M}_0D^{-1}\left(W^{-1,\tau}\{\mathbf{r}_x\}\right)^\top w_0+\\ +w_0^{-1}\Phi\mathcal{M}D^{-1}\Psi^\top w_0\end{gathered}\tag{44}$$

We shall point out that: $\Psi^\top w_0=\Delta^{-1}\psi^\top(I-\varphi\Delta^{-1}D^{-1}\{\psi^\top\})=\left(\Delta^{-1}D^{-1}\{\psi^\top\}\right)_x=\tilde{\Psi}_x^\top$. We shall also observe that:

$$\begin{gathered}(W^{-1,\tau}\{\mathbf{r}_x\})^\top w_0=\left(\mathbf{r}_x^\top-D^{-1}\{\mathbf{r}_x^\top\varphi\}\Delta^{-1}\psi^\top\right)\left(I-\varphi\Delta^{-1}D^{-1}\{\psi^\top\}\right)=\\ \left(\mathbf{r}^\top-D^{-1}\{\mathbf{r}_x^\top\varphi\}\Delta^{-1}D^{-1}\{\psi^\top\}\right)_x=(D^{-1}W_m^{-1,\tau}D\{\mathbf{r}\})_x^\top=\tilde{\mathbf{r}}_x^\top\end{gathered}$$

Thus (44) can be rewritten as:

$$\begin{gathered}\tilde{L}_k=W_mL_kW_m^{-1}=w_0^{-1}W\left(\beta_k\partial_{\tau_k}+B_k-\mathbf{q}\mathcal{M}_0\mathbf{r}^\top+\mathbf{q}\mathcal{M}D^{-1}\mathbf{r}_x^\top\right)W^{-1}w_0=\\ \beta_k\partial_{\tau_k}+(W_mL_kW_m^{-1})_{\geq 0}+\tilde{\mathbf{q}}\mathcal{M}_0D^{-1}\tilde{\mathbf{r}}_x^\top-\tilde{\Phi}\mathcal{M}D^{-1}\tilde{\Psi}_x^\top=\\ \beta_k\partial_{\tau_k}+(W_mL_kW_m^{-1})_{\geq 0}+\tilde{\mathbf{q}}\mathcal{M}_0\tilde{\mathbf{r}}^\top-\tilde{\Phi}\mathcal{M}\tilde{\Psi}^\top-\tilde{\mathbf{q}}\mathcal{M}_0D^{-1}\tilde{\mathbf{r}}^\top D-\\ +\tilde{\Phi}\mathcal{M}D^{-1}\tilde{\Psi}^\top D\end{gathered}\tag{45}$$

Using that $\tilde{L}_k\{1\}=\tilde{u}_0=0$ we obtain the form of $\tilde{B}_k$. I.e., $\tilde{B}_k:=(W_mL_kW_m^{-1})_{\geq 0}+\tilde{\mathbf{q}}\mathcal{M}_0\tilde{\mathbf{r}}^\top-\tilde{\Phi}\mathcal{M}\tilde{\Psi}^\top=\sum_{j=1}^k\tilde{u}_jD^j$. □

Theorem 2 provides us with a dressing method for k-cmKP hierarchy (11). I.e., the following corollary directly follows from the previous theorem:

Corollary 1. *Assume that operators L_k and M_n in (11) satisfy Lax equation: $[L_k,M_n]=0$. Let functions φ and ψ satisfy equations:*

$$\begin{gathered}L_k\{\varphi\}=\varphi\Lambda,\ L_k^\tau\{\psi\}=\psi\tilde{\Lambda},\ \Lambda,\tilde{\Lambda}\in Mat_{K\times K}(\mathbb{C}),\\ M_n\{\varphi\}=0,\ M_n^\tau\{\psi\}=0.\end{gathered}\tag{46}$$

Then transformed operators $\tilde{L}_k=W_mL_kW_m^{-1}$ (see (42) with $\beta_k=0$) and

$$\tilde{M}_n=W_mM_nW_m^{-1}=\alpha_n\partial_{t_n}-D^n-\sum_{i=1}^{n-1}\tilde{u}_iD^i\tag{47}$$

via transformation W_m (41) also satisfy Lax equation: $[\tilde{L}_k,\tilde{M}_n]=0$

Proof. It can be checked directly that: $[\tilde{L}_k,\tilde{M}_n]=[W_mL_kW_m^{-1},W_mM_nW_m^{-1}]=W_m[L_k,M_n]W_m^{-1}=0$. The exact form of operators $\tilde{L}_k$ and $\tilde{M}_n$ follows from Theorem 2. □

The following corollary follows from Corollary 1 and Theorem 2:

Corollary 2. *Suppose that functions φ and ψ satisfy equations (46) with operators L_k and M_n defined by (23) then transformed operators have the form:*

$$\begin{gathered}\tilde{L}_k=B_k-\tilde{\mathbf{q}}\mathcal{M}_0D^{-1}\tilde{\mathbf{r}}^\top D+\tilde{\Phi}\mathcal{M}D^{-1}\tilde{\Psi}^\top D+\tilde{v}+\beta D^{-1}\tilde{u},\\ \tilde{M}_n=\alpha_n\partial_{t_n}-\tilde{B}_n,\ \tilde{B}_n=D^n+\textstyle\sum_{i=1}^{n-1}\tilde{u}_iD^i,\end{gathered}\tag{48}$$

where

$$
\begin{array}{l}
\mathcal{M} = C\Lambda - \tilde{\Lambda}^{\top}C,\ \tilde{\Phi} = -W_m\{\varphi\}C^{-1} = \varphi\tilde{\Delta}^{-1},\\
\tilde{\Psi} = D^{-1}\{W_m^{\tau,-1}\{\psi\}\}C^{-1,\top} = D^{-1}\{\psi\}\Delta^{-1,\top},\ \tilde{\mathbf{q}} = W_m\{\mathbf{q}\},\\
\tilde{\mathbf{r}} = W_m^{-1,\tau}\{\mathbf{r}\}, \tilde{\Delta} = -C + D^{-1}\{D^{-1}\{\psi^{\top}\}\varphi_x\}, \Delta = C + D^{-1}\{\psi^{\top}\varphi\},\\
\tilde{u} = W_m^{-1,\tau}\{D^{-1}\{u\}\},\ \tilde{v} = W_m\{v\} + \beta D^{-1}W_m^{-1,\tau}\{u\} - \beta W_m\{D^{-1}\{u\}\}.
\end{array}
\tag{49}
$$

As it was shown in previous Sections the most interesting systems arise from the k-cmKP hierarchy (11) and its reduction (23) after a Hermitian conjugation reduction. Our aim is to show that under additional restrictions Binary Darboux Transformation W_m (41) preserves this reduction.

Proposition 1. (1) *Let $\psi = \bar{\varphi}_x$ and $C = -C^*$ in the dressing operator W_m (41). Then the operator W_m is D-unitary ($W_m^{-1} = D^{-1}W_m^*D$).*

(2) *Let the operator L_k (11) be D-Hermitian: $L_k^* = DL_kD^{-1}$ (D-skew-Hermitian: $L_k^* = -DL_kD^{-1}$) and M_n (11) be D-Hermitian (D-skew-Hermitian). Then the operator $\hat{L}_k = W_mL_kW_m^{-1}$ (see (42)) transformed via the D-unitary operator W_m is D-Hermitian (D-skew-Hermitian) and $\hat{M}_n := W_mM_nW_m^{-1}$ (47) is D-Hermitian (D-skew-Hermitian).*

(3) *Assume that the conditions of items 1 and 2 hold. Let $\tilde{\Lambda} = \bar{\Lambda}$ in the case of D-Hermitian L_k ($\tilde{\Lambda} = -\bar{\Lambda}$ in D-skew-Hermitian case). We shall also assume that the function φ satisfies the corresponding equations in formulae (46). Then $\mathcal{M} = \mathcal{M}^*$ ($\mathcal{M} = -\mathcal{M}^*$) and $\tilde{\Psi} = \bar{\tilde{\Phi}}$ (see formulae (39)).*

In subparagraph 4.1 we will show how one can use methods described in Theorem 2 and its corollaries in order to obtain solutions of KdV equation (33) and its generalization (30)

4.1. **Solution generating technique for system (30) and KdV equation (33).** We shall consider equation (30) in case the dimension of vector $\mathbf{q}$ and matrix $\mathcal{M}_0$ is even. I.e., $m = 2\tilde{m}$, $\tilde{m} \in \mathbb{N}$ (in this situation skew-symmetric matrix $\mathcal{M}_0$ can be non-degenerate). Assume that the skew-symmetric matrix $\mathcal{M}_0$ in (30) and vector-function $\mathbf{q}$ has the form:

$$
\mathcal{M}_0 = \begin{pmatrix} 0_{\tilde{m}} & I_{\tilde{m}} \\ -I_{\tilde{m}} & 0_{\tilde{m}} \end{pmatrix},\ \mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2) = (q_{11}, q_{12}, \ldots, q_{1\tilde{m}}, q_{21}, q_{22}, \ldots, q_{2\tilde{m}}),
\tag{50}
$$

where $0_{\tilde{m}}$ is a $\tilde{m}\times\tilde{m}$-dimensional matrix consisting of zeros, $I_{\tilde{m}}$ is an identity matrix with the dimension $\tilde{m}\times\tilde{m}$. Equation (30) after notation $\tilde{u} := u$ can be rewritten in the following form:

$$
\begin{array}{r}
\alpha_3\mathbf{q}_{1,t_3} = \mathbf{q}_{1,xxx} - 3(\mathbf{q}_{1,x}\mathbf{q}_2^{\top} - \mathbf{q}_{2,x}\mathbf{q}_1^{\top})\mathbf{q}_{1,x} + 3\beta\tilde{u}\mathbf{q}_{1,x},\\
\alpha_3\mathbf{q}_{2,t_3} = \mathbf{q}_{2,xxx} - 3(\mathbf{q}_{1,x}\mathbf{q}_2^{\top} - \mathbf{q}_{2,x}\mathbf{q}_1^{\top})\mathbf{q}_{2,x} + 3\beta\tilde{u}\mathbf{q}_{2,x},\\
\alpha_3\tilde{u}_{t_3} = \tilde{u}_{xxx} - 3(\tilde{u}(\mathbf{q}_{1,x}\mathbf{q}_2^{\top} - \mathbf{q}_{2,x}\mathbf{q}_1^{\top}))_x + 6\beta\tilde{u}\tilde{u}_x.
\end{array}
\tag{51}
$$

In this subsection our aim is to consider the case $\tilde{m} = 1$ (although the corresponding solution generating technique can be generalized to the case of an arbitrary natural $\tilde{m}$). In this situation $\mathbf{q}_1 = q_1$ and $\mathbf{q}_2 = q_2$ are scalars. We shall suppose that $K = 2\tilde{K}$ is an even natural number. Assume that the function φ is $(1\times K)$-vector solution of the system:

$$
\begin{array}{c}
L_{10}\{\varphi\} = \varphi_x + \beta D^{-1}\{u\varphi\} = \varphi\Lambda,\ \Lambda \in Mat_{K\times K}(\mathbb{C}),\ \beta \in \mathbb{R},\\
M_{30}\{\varphi\} = \alpha_3\varphi_{t_3} - \varphi_{xxx} - 3\beta u\varphi_x = 0,
\end{array}
\tag{52}
$$

with a number $u \in \mathbb{R}$.

Using Theorem 2 and Proposition 1 we obtain that dressed operators $\tilde{L}_{10}$ and $\tilde{M}_{30}$ via operator W_m (41) with skew-Hermitian matrix C and $\psi = \bar{\varphi}_x$ has the form:

$$\begin{array}{l} \tilde{L}_{10} = W_m L_{10} W_m^{-1} = D + \tilde{\Phi}\mathcal{M}D^{-1}\tilde{\Phi}^* D + \beta D^{-1}\tilde{u} + \tilde{v} \\ \tilde{M}_{30} = W_m M_{30} W_m^{-1} = \alpha_3\partial_{t_3} - D^3 - (\tilde{v} + \tilde{\Phi}\mathcal{M}\tilde{\Phi}^*)D^2 - \\ -3\left((\tilde{\Phi}\mathcal{M}\tilde{\Phi}^* + \tilde{v})^2 + \tilde{\Phi}_x\mathcal{M}\tilde{\Phi}^* + \tilde{v}_x + \beta\tilde{u}\right)D, \end{array} \tag{53}$$

where $\mathcal{M} = C\Lambda - \Lambda^* C^*$, $\tilde{\Phi} = \varphi\tilde{\Delta}^{-1}$, $\tilde{u} = u - D\{\bar{\varphi}\bar{\tilde{\Delta}}^{-1}D^{-1}\{\varphi^\top u\}\}$, $\tilde{v} = \beta(\tilde{\Phi}D^{-1}\{\varphi^* u\} - D^{-1}\{u\varphi\}\tilde{\Phi}^*)$, $\tilde{\Delta} = -C + D^{-1}\{\varphi^*\varphi_x\}$. It has to be pointed out that the function $\tilde{\Phi} = -W_m\{\varphi\}C^{-1} = \varphi\tilde{\Delta}^{-1}$ satisfies equation: $\tilde{M}_{30}\{\tilde{\Phi}\} = 0$ because $\tilde{M}_{30}\{\tilde{\Phi}\} = W_m M_{30} W_m^{-1}\{W_m\{\varphi\}C^{-1}\} = 0$.

Now we assume that function φ, matrices C and Λ are real. In this case $\tilde{v} = \tilde{v}^\top = \beta(\tilde{\Phi}D^{-1}\{\varphi^\top u\} - D^{-1}\{u\varphi\}\tilde{\Phi}^\top)^\top = -\tilde{v} = 0$.

Let us put

$$\Lambda = diag(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \dots, \lambda_{\tilde{K}1}, \lambda_{\tilde{K}2}),\ \lambda_{ij} \in \mathbb{R},$$

$$C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1\tilde{K}} \\ C_{21} & C_{22} & \dots & C_{2\tilde{K}} \\ \vdots & \vdots & \dots & \vdots \\ C_{\tilde{K}1} & C_{\tilde{K}2} & \dots & C_{\tilde{K}\tilde{K}} \end{pmatrix}, \tag{54}$$

where elements C_{ij} are (2×2)-matrices of the form:

$$C_{ij} = \begin{pmatrix} 0 & -\frac{1}{\lambda_{j2}+\lambda_{i1}} \\ \frac{1}{\lambda_{j1}+\lambda_{i2}} & 0 \end{pmatrix}. \tag{55}$$

Under such a choice of C (55) and Λ (4.1) we obtain that $2\tilde{K}\times 2\tilde{K}$-dimensional matrix $\mathcal{M} = C\Lambda - \Lambda^\top C^\top$ has the block form: $\mathcal{M} = (\mathcal{M}_{ij})_{i,j=1}^{\tilde{K}}$, where $\mathcal{M}_{ij} = \mathcal{M}_0$ (see formula (50) in case $\tilde{m} = 1$). Let us denote by: $\mathbf{1}_{\tilde{K}} = (I_2, \dots, I_2)$ matrix that consists of $\tilde{K}$ (2×2)-dimensional identity matrices I_2. Then $\mathcal{M} = -\mathbf{1}_{\tilde{K}}^\top\mathcal{M}_0\mathbf{1}_{\tilde{K}}$.

Let us put $u = const$ and choose solution of system (52) in the form: $\varphi = (\varphi_{11}, \varphi_{12}\varphi_{21}, \varphi_{22}, \dots, \varphi_{\tilde{K}1}, \varphi_{\tilde{K}2})$, $\varphi_{ij} = exp\left\{(\frac{1}{2}\lambda_{ij} + \gamma_{ij})x + a_{ij}t\right\}$, where $\gamma_{ij} = \sqrt{\frac{1}{4}\lambda_{ij}^2 - \beta u}$, $a_{ij} = \left\{\left(\frac{1}{2}\lambda_{ij} + \gamma_{ij}\right)^3 + 3\beta u\left(\frac{1}{2}\lambda_{ij} + \gamma_{ij}\right)\right\}/\alpha_3$. $(2\tilde{K}\times 2\tilde{K})$-matrix $\tilde{\Delta}$ then takes the block form:

$$\tilde{\Delta} = -C + D^{-1}\{\varphi^\top\varphi_x\} = \left(\tilde{\Delta}_{ij}\right)_{i,j=1}^{\tilde{K}} =$$

$$= \begin{pmatrix} \frac{\alpha_{i1}}{\alpha_{i1}+\alpha_{j1}}e^{(\alpha_{i1}+\alpha_{j1})x+(a_{i1}+a_{j1})t} & \frac{\alpha_{i2}}{\alpha_{i2}+\alpha_{j1}}e^{(\alpha_{i2}+\alpha_{j1})x+(a_{i2}+a_{j1})t} + \frac{1}{\lambda_{j2}+\lambda_{i1}} \\ \frac{\alpha_{i1}}{\alpha_{i1}+\alpha_{j2}}e^{(\alpha_{i1}+\alpha_{j2})x+(a_{i1}+a_{j2})t} - \frac{1}{\lambda_{j1}+\lambda_{i2}} & \frac{\alpha_{i2}}{\alpha_{i2}+\alpha_{j2}}e^{(\alpha_{i2}+\alpha_{j2})x+(a_{i2}+a_{j2})t} \end{pmatrix}_{i,j=1}^{\tilde{K}}, \tag{56}$$

where $\alpha_{ij} = \frac{1}{2}\lambda_{ij} + \gamma_{ij}$. Functions $\mathbf{q} = (q_1, q_2) = \varphi\tilde{\Delta}^{-1}\mathbf{1}_{\tilde{K}}^\top$ and $\tilde{u} = u - D\left\{\varphi\tilde{\Delta}^{-1}D^{-1}\{\varphi^\top u\}\right\}$ will be solutions of system (51).

We shall point out that in case $\beta = 0$, $\tilde{K} = 1$, $\alpha_3 = 1$ we obtain the following solution of the real version of the mKdV-type equation (equation (51) with $\tilde{u} = 0$):

$$\begin{array}{l} \mathbf{q} = (q_1, q_2), \quad q_1 = -\frac{2(\lambda_{11}+\lambda_{12})\varphi_{12}}{(\lambda_{11}-\lambda_{12})\varphi_{11}\varphi_{12}-2}, \quad q_2 = \frac{2(\lambda_{11}+\lambda_{12})\varphi_{11}}{(\lambda_{11}-\lambda_{12})\varphi_{11}\varphi_{12}-2}, \\ \varphi_{1j} = e^{\lambda_{1j}x+\lambda_{1j}^3 t_3},\ \lambda_{1j} > 0,\ j = \overline{1,2}. \end{array}$$

It is also possible to choose other types of matrices C and Λ in (4.1) and (55). In particular the following remark holds:

Remark 1. *In case $\tilde{K}=1$ vector of functions $\varphi=(\varphi_1,\varphi_2)$, $\varphi_1=\cos(x\lambda_{12}+(3\lambda_{11}^2\lambda_{12}-\lambda_{12}^3)t+\frac{\pi}{4})e^{x\lambda_{11}+(\lambda_{11}^3-3\lambda_{11}\lambda_{12}^2)t}$, $\varphi_2=\sin(x\lambda_{12}+(3\lambda_{11}^2\lambda_{12}-\lambda_{12}^3)t+\frac{\pi}{4})e^{x\lambda_{11}+(\lambda_{11}^3-3\lambda_{11}\lambda_{12}^2)t}$. will be a solution of the system (52) with $u=0$ and $\Lambda=\begin{pmatrix}\lambda_{11} & \lambda_{12}\\ -\lambda_{12} & \lambda_{11}\end{pmatrix}$. The corresponding solution generating technique given by (4.1)-(4.1) in case $\tilde{K}=1$, $C_{\tilde{K}}=C_1=\begin{pmatrix}0 & \frac{1}{2\lambda_{11}}\\ -\frac{1}{2\lambda_{11}} & 0\end{pmatrix}$ gives us a solution of mKdV-type equation (51) with $\tilde{u}=0$ that coincides with a solution obtained in [36].*

Now we will consider solution generating technique for KdV (33). For this purpose we assume that function φ, matrices $\Lambda=diag(\Lambda_1,\dots,\Lambda_{\tilde{K}})$ and $C=diag(C_1\dots,C_{\tilde{K}})$ are real and have the form:

$$\Lambda_j=\begin{pmatrix}0 & \lambda_j\\ \lambda_j & 0\end{pmatrix},\ C_j=\begin{pmatrix}0 & -c_j\\ c_j & 0\end{pmatrix}. \tag{57}$$

In this case we obtain that the matrix $\mathcal{M}=C\Lambda-\Lambda^\top C^\top$ consists of zeros in (53). Consider the following solution of system (52):

$$\varphi=(\varphi_{11},\varphi_{12},\varphi_{21},\varphi_{22},\dots,\varphi_{\tilde{K}1},\varphi_{\tilde{K}2}),$$

$$\varphi_{j1}=e^{\gamma_j x+a_j t}\cosh\left(\frac{\lambda_j}{2}x+b_j t\right),\ \varphi_{j2}=e^{\gamma_j x+a_j t}\sinh\left(\frac{\lambda_j}{2}x+b_j t\right),$$

where $\gamma_j=\sqrt{\frac{1}{4}\lambda_j^2-\beta u}$, $a_j=\left(\gamma_j^3+\frac{3}{4}\gamma_j\lambda_j^2+3\beta u\gamma_j\right)/\alpha_3$, $b_j=\left(3\gamma_j^2\frac{\lambda_j}{2}+\frac{\lambda_j^3}{8}+\frac{3}{2}\beta u\lambda_j\right)/\alpha_3$ and λ_j, α_3, β, $u\in\mathbb{R}$. Thus, $\tilde{v}=0$ and we obtain Lax pair for KdV equation in (53): $\tilde{L}_{10}=D+\beta D^{-1}\tilde{u}$, $\tilde{M}_{30}=\alpha_3\partial_{t_3}-D^3-3\beta\tilde{u}D$.

Formula

$$\tilde{u}=u-D\left\{\varphi\tilde{\Delta}^{-1}D^{-1}\{\varphi^\top u\}\right\}:=u+\hat{u}, \tag{58}$$

gives us a finite density solution of equation (33). In particular, if $\tilde{K}=1$ and $c_1=\frac{1}{8}\frac{\lambda_1}{\gamma_1}$ we obtain the following solution:

$$\tilde{u}=u+\frac{2\gamma_1^2}{\beta\cosh^2(\gamma_1 x+a_1 t)}. \tag{59}$$

Now we shall substitute $\tilde{u}$ (58) in KdV equation (33):

$$\alpha_3\hat{u}_{t_3}=\hat{u}_{xxx}+6\beta\hat{u}\hat{u}_x+6\beta u\hat{u}_x. \tag{60}$$

The corresponding pair of operators have the form: $L_1=D+\beta D^{-1}(\hat{u}+u)$, $M_3=\alpha_3\partial_{t_3}-D^3-3\beta\hat{u}D-3\beta uD$. We have two ways to obtain soliton solutions (that are rapidly decreasing at both infinities in contradistinction to finite density solutions (58) that tend to an arbitrary real number u) for KdV from formula (58):

(1) By taking the limit $u\to 0$ in (58)-(60).
(2) By making a change of the independent variables: $\tilde{x}:=x+6\alpha_3^{-1}\beta u t_3$, $\tilde{t}_3:=t_3$ and $\hat{v}(\tilde{x},\tilde{t}_3):=\hat{u}(x,t_3)$ in equation (60) and solutions (58)-(59). This change corresponds to the change of differential operators in the Lax pair for equation (60) consisting of operators L_1 and M_3: $\alpha_3\partial_{\tilde{t}_3}=\alpha_3\partial_{t_3}-3\beta uD$.

5. Conclusions

In this paper we obtain new generalizations (23) of the modified k-cKP (k-cmKP) hierarchy (11). The obtained hierarchy also generalizes the BKP hierarchy [36–38] which is the special case of the k-cmKP hierarchy. Dressing methods elaborated via BDT-type operators (Section 4) give rise to exact solutions of the integrable systems that hierarchy (23) contains. In particular, soliton solutions for generalization of mKdV-type equation (51) and finite density solutions as well as regular soliton solutions were constructed for the KdV equation using the proposed dressing methods. This methods also allow to obtain rational and singular multi-soliton solutions of the corresponding nonlinear systems under the special choice of spectral matrix Λ in the linear system (52). In order to minimize the size of this article we do not include those results here. We shall point out that the special case of equation (51) ($\tilde{u}=0$) and its solutions were considered in [36]. Generalizations (23) of the k-cmKP hierarchy (11) together with different extensions of k-cKP hierarchy (including (1+1) and (2+1)-BDk-cKP hierarchy [32–34]) is a good basis for construction of other hierarchies of nonlinear integrable equations with corresponding dressing methods. In particular in our forthcoming papers we plan to introduce (2+1)-BDk-cmKP hierarchy and investigate solution generating technique for the corresponding integrable systems. Consider as an example Lax pair from the (1+1)-BDk-cKP hierarchy that was investigated in [33]:

$$\begin{aligned}
&P_{1,1} = D + c_1 M_2\{\mathbf{q}\}\mathcal{M}_0 D^{-1}\mathbf{r}^\top + c_1\mathbf{q}\mathcal{M}_0 D^{-1}(M_2^\tau\{\mathbf{r}\})^\top + c_0\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top = \\
&= D + c_1\big(\alpha_2\mathbf{q}_{t_2}\mathcal{M}_0 D^{-1}\mathbf{r}^\top - \alpha_2\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}_{t_2}^\top - \mathbf{q}_{xx}\mathcal{M}_0 D^{-1}\mathbf{r}^\top - \\
&-\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}_{xx}^\top - u\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top u\big) + c_0\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top, \\
&M_2 = \alpha_2\partial_{t_2} - D^2 - u.
\end{aligned} \tag{61}$$

It was shown in [33] that the Lax equation $[P_{1,1}, M_2]=0$ in (61) is equivalent to the system:

$$[P_{1,1}, M_2]_{\geq 0} = 0,\ c_1 M_2^2\{\mathbf{q}\} + c_0 M_2\{\mathbf{q}\} = 0,\ c_1(M_2^\tau)^2\{\mathbf{r}\} + c_0 M_2^\tau\{\mathbf{r}\} = 0. \tag{62}$$

that is equivalent to the generalization of the AKNS system. In case $c_0 = 1$, $c_1 = 0$ we obtain AKNS system in (62):

$$\alpha_2\mathbf{q}_{t_2} - \mathbf{q}_{xx} - u\mathbf{q} = 0,\ -\alpha_2\mathbf{r}_{t_2} - \mathbf{r}_{xx} - u\mathbf{r} = 0,\ u = 2\mathbf{q}\mathcal{M}_0\mathbf{r}^\top.$$

Assume that the scalar function f satisfies equations $P_{1,1}\{f\} = f\lambda$, $M_2\{f\} = 0$. We shall introduce the notations $\tilde{M}_2 := f^{-1}M_2 f$, $\hat{M}_2 := D\tilde{M}_2 D^{-1}$, $\tilde{P}_{1,1} := f^{-1}P_{1,1}f$, $\tilde{\mathbf{q}} := f^{-1}\mathbf{q}$, $\tilde{\mathbf{r}}^\top := D^{-1}\{\mathbf{r}^\top f\}$ and consider the following gauge transformations

$$\begin{aligned}
&\tilde{M}_2 = f^{-1}M_2 f = \alpha_2\partial_{t_2} - D^2 - 2\tilde{u}D,\ \tilde{u} = f^{-1}f_x, \\
&\tilde{P}_{1,1,} = f^{-1}P_{1,1}f = D + f^{-1}f_x + c_1 f^{-1}M_2\{\mathbf{q}\}\mathcal{M}_0 D^{-1}\mathbf{r}^\top f + \\
&+c_1 f^{-1}\mathbf{q}\mathcal{M}_0 D^{-1}(M_2^\tau\{\mathbf{r}\})^\top f + c_0 f^{-1}\mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top f = \\
&= D - c_1\tilde{M}_2\{\tilde{\mathbf{q}}\}\mathcal{M}_0 D^{-1}\tilde{\mathbf{r}}^\top D - c_1\tilde{\mathbf{q}}\mathcal{M}_0 D^{-1}(\hat{M}_2^\tau\{\tilde{\mathbf{r}}\})^\top D - c_0\tilde{\mathbf{q}}\mathcal{M}_0 D^{-1}\tilde{\mathbf{r}}^\top D.
\end{aligned}$$

The equation $[\tilde{M}_2, \tilde{P}_{1,1}] = 0$ is equivalent to the following system:

$$[\tilde{P}_{1,1}, \tilde{M}_2]_{>0} = 0,\ c_1\tilde{M}_2^2\{\tilde{\mathbf{q}}\} + c_0\tilde{M}_2\{\tilde{\mathbf{q}}\} = 0,\ c_1(\hat{M}_2^\tau)^2\{\tilde{\mathbf{r}}\} + c_0\hat{M}_2^\tau\{\tilde{\mathbf{r}}\} = 0. \tag{63}$$

or in the equivalent form (after notation $\mathbf{q}_0 := \tilde{\mathbf{q}}$, $\mathbf{r}_0 := \tilde{\mathbf{r}}$):

$$\begin{aligned}
&[\tilde{P}_{1,1}, \tilde{M}_2]_{>0} = 0,\ \mathbf{q}_1 = \tilde{M}_2\{\mathbf{q}_0\},\ \mathbf{r}_1 = \hat{M}_2^\tau\{\mathbf{r}_0\}, \\
&c_1\tilde{M}_2\{\mathbf{q}_1\} + c_0\tilde{M}_2\{\mathbf{q}_0\} = 0,\ c_1\hat{M}_2^\tau\{\mathbf{r}_1\} + c_0\hat{M}_2^\tau\{\mathbf{r}_0\} = 0.
\end{aligned} \tag{64}$$

System (64) is the generalization of the Chen-Lee-Liu system (case $c_1 = 0$, $c_0 = 1$). In case of additional reduction $\alpha_2 \in i\mathbb{R}$, $c_0 = 0$, $c_1 \in \mathbb{R}$, $\mathcal{M}_0^* = -\mathcal{M}_0$, $\mathbf{r} = \bar{\mathbf{q}}$ (64) reads as following:

$$\begin{aligned} &\alpha_2 \mathbf{q}_{0,t_2} - \mathbf{q}_{0,xx} + 2c_1(\mathbf{q}_1 \mathcal{M}_0 \mathbf{q}_0^* + \mathbf{q}_0 \mathcal{M}_0 \mathbf{q}_1^*)\mathbf{q}_{0,x} - \mathbf{q}_1 = 0, \\ &\alpha_2 \mathbf{q}_{1,t_2} - \mathbf{q}_{1,xx} + 2c_1(\mathbf{q}_1 \mathcal{M}_0 \mathbf{q}_0^* + \mathbf{q}_0 \mathcal{M}_0 \mathbf{q}_1^*)\mathbf{q}_{1,x} = 0. \end{aligned} \tag{65}$$

We shall also point out that the extension of the k-cmKP hierarchy (23) can also be generalized to the matrix case. It leads to matrix generalizations of integrable systems that hierarchy (23) contains (including Chen-Lee-Liu (16) and modified-type KdV equation (19)). In particular, the matrix generalization of the modified KdV-type equation (19) differs from the well-known matrix mKdV equation that was investigated by the inverse scattering method in [39].

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Ivan Franko National University of L'viv, 1 Universytetska st., 79000 L'viv, Ukraine
E-mail address: `alex.chvartatskyy@gmail.com, y_sydorenko@franko.lviv.ua`